ON THE FORMULATION OF THE PROBLEM OF IMPACT OF A VISCOUS-PLASTIC BAR

(O POSTANOVKE ZADACHI OB UDARE VIAZKO-PLASTICHESKOGO STERZHNIA)

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The impact of a viscous-plastic bar against a rigid abstacle was considered in [1] . It was assumed that the plastic deformations of the bar were confined to the region of impact, with the rest of the bar moving as a rigid body. Separately, it was assumed that the boundary between the plastic and the rigid regions moved from the point of impact towards the free end. This natural assumption has recently aroused some doubt, chiefly on the grounds of the inherent implication that a bar of infinite length would become loaded and plastic along its entire length immediately upon impact against the rigid obstacle. It was concluded that upon the impact of a bar of finite length, the plastic region should spread instantaneously over the entire length, with the boundary propagating from the free end.

It will be shown in this paper that the boundary actually originates at the end of impact and that when $l \to \infty$ (l is the length of the bar), the average velocity of transition of the boundary tends to infinity over any finite portion of the bar.

1. We shall use a Lagrange system of coordinates, moving together with the obstacle with velocity v_0 in such a way that at the initial instant the bar is at rest, with the x-axis directed along the bar, the obstacle is at $x = 0$. We shall adopt the notation: $v(t, x)$ is the velocity, $\sigma(t, x)$ is the axial stress, t is time, l is the length of the bar. We shall use the equations of state in the form

$$\frac{\partial v}{\partial x} = 0 \qquad \text{for } |\sigma| \leqslant |\sigma_0| \tag{1.1}$$

$$\frac{\partial v}{\partial x} = \mu(\sigma - \sigma_0)|\sigma - \sigma_0|^{\alpha} \qquad \text{for } |\sigma| > |\sigma_0| \tag{1.2}$$

We shall assume the following: in the plane xt there exists a curve $x = x_0 t$ which divides the region $t \geqslant 0$, $l \geqslant x \geqslant 0$ into regions D_1 and D_2 (Fig.1); the velocity $v(t, x)$ in D_1 satisfies (1.2) and the equation of motion

$$\rho_0 v_t = \sigma_x \tag{1.3}$$

together with the boundary conditions (1.4)

$$v(t, 0) = v_0, \qquad v(0, x) = 0, \qquad v_x(t, x_0(t)) = 0$$

and the velocity $v(t, x)$ in D_2 satisfies

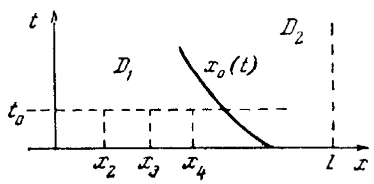

Fig. 1

(1.1) and the equation of motion

$$\rho_0 \frac{dv}{dt} = -\frac{\sigma_0}{l - x_0(t)} \tag{1.5}$$

with the initial condition $v(0, x) = 0$.

The function $v(t, x)$ is continuous for all $l \geqslant x \geqslant 0$, $t \geqslant 0$, apart from point $x = 0$, $t = 0$. The unknown functions are $v(t, x)$, $\sigma(t, x)$ and $x_0(t)$.

We shall further stipulate that $v \geqslant 0, v_0 > 0, \sigma < 0, \sigma_0 < 0$. The assumptions of [1] correspond to $\sigma = 0$, $x_0(0) = 0$.

2. We shall now prove that $x_0(0) = 0$. Equations (1.2) and (1.3), together with the boundary conditions (1.4), fully define for region D_1 the function $v(t, x)$ which depends upon $x_0(t)$ and is in a particular case explicitly defined by

$$v_-(t) = \lim v(t, x) \qquad \text{for} \quad x \to x_0(t) - 0$$

Similarly, for region D_2, Equation (1.5) explicitly defines $v(t, x)$, dependent upon $x_0(t)$, as well as $v_+(t) = \lim v(t, x)$ when $x \to x_0(t) + 0$. By virtue of the continuity of $v(t, x)$ over $t \geqslant 0$, $l \geqslant x \geqslant 0$ (apart from point $t = 0$, $x = 0$), we have

$$v_-(t) = v_+(t) \tag{2.1}$$

which must be satisfied by a suitable choice of $x_0(t)$. We shall show that if $x_0(0) > 0$, then (2.1) is not satisfied. To do so, we shall assume that $x_0(0) > 0$, we estimate the upper limit of $v_-(t)$ and the lower of $v_+(t)$ and prove that as $t \to 0$ the upper estimate is below the lower; that is, $v_-(t) < v_+(t)$ which contradicts (2.1).

Let us first evaluate $v_+(t)$. From (1.1) and (1.5) we have

$$v_1(t) = v(t, x) = \frac{1}{\rho_0} \int_0^t \frac{|\sigma_0|}{l - x_0(\xi)} d\xi \geqslant \frac{|\sigma_0|}{\rho_0 l} t \tag{2.2}$$

Since $x_0 \leqslant l$, this equation holds also for $x_0(0) = l$.

To evaluate $v_-(t)$ we shall use the condition $v_x < 0$ in D_1; this condition follows from $\sigma < 0$ and (1.2). The quantity $\sigma(t, x)$ is defined explicitly by Equations (1.2) through (1.4), and the assumption that $\sigma < 0$ may not always be valid; but the question of solvability of the problem as a whole is beyond the scope of the present argument.

Thus we shall estimate $v_-(t)$ on the assumption that $x_0(0) > 0$. If this were true, then the inequality $t_0 > 0$ would also hold and x_2, x_3 and x_4 would satisfy $x_0(t) > x_4 > x_3 > x_2 > 0$, for $t_0 \geqslant t \geqslant 0$ and would otherwise be arbitrary. We shall prove that

$$\int_{x_3}^{x_4} v(t, x)\, dx = 0(1)\, t \tag{2.3}$$

where $0(1) \to 0$ when $t \to 0$. Taking into account (1.3) as well as $\sigma < 0$, $|\sigma| > |\sigma_0|$, we have

$$\rho_0 \int_{x_3}^{x_4} v(t, x)\, dx = \int_{x_3}^{x_4} \int_0^t \frac{\partial [\sigma(s, x) - \sigma_0]}{\partial x} ds\, dx = \int_0^t [\sigma(s, x_4) - \sigma_0]\, ds - \int_0^t [\sigma(s, x_3) - \sigma_0]\, ds \leqslant$$

$$\leqslant -\int_0^t [\sigma(s, x_3) - \sigma_0]\, ds \leqslant -\frac{1}{x_3 - x_2} \int_{x_2}^{x_3} \int_0^t [\sigma(s, x) - \sigma_0]\, ds\, dx$$

here we have made use of

$$\frac{\partial}{\partial x} \int_0^t [-\sigma(s, x) + \sigma_0]\, ds = -\rho_0 v(t, x) < 0$$

Using (1.2) and Hölder's inequality (which here holds also for $\alpha = 0$), we obtain

$$\int_{x_2}^{x_3}\int_0^t [-\sigma(s,x)+\sigma_0]\,ds\,dx \leqslant [(x_3-x_2)t]^{\frac{\alpha}{1+\alpha}}\left(\int_{x_2}^{x_3}\int_0^t [-\sigma(s,x)+\sigma_0]^{1+\alpha}\,ds\,dx\right)^{\frac{1}{1+\alpha}} =$$

$$= [(x_3-x_2)\,t]^{\frac{\alpha}{1+\alpha}}\left(\frac{1}{\mu}\int_0^t [v(s,x_2)-v(s,x_3)]\,ds\right)^{\frac{1}{1+\alpha}} = [(x_3-x_2)\,t]^{\frac{\alpha}{1+\alpha}}[O(1)\,t]^{\frac{1}{1+\alpha}} = O(1)\,t$$

which proves (2.3). Since $v_x \leqslant 0$

$$v_-(t) \leqslant v(t,x_3) \leqslant \frac{1}{x_4-x_3}\int_{x_3}^{x_4} v(t,x)\,dx = 0(1)\,t$$

that is $v_-(t) = O(1)t$. In the light of (2.2) this gives $v_-(t) < v_+(t)$ at sufficiently small values of t which contradicts (2.1) and thus proves that $x_0(0) > 0$ is inadmissible. It is worth noting that this proof is inapplicable for $l = \infty$ or $|\sigma_0| = 0$, since in such cases the right-hand side of (2.2) becomes zero.

3. We shall next assume that $\alpha = 0$ and introduce the function $\tau(x)$, the inverse of $x_0(t)$, that is $\tau(x_0(t)) = t$. For this function we obtain the estimate

$$\tau(x) \leqslant x^2 H\left(\frac{|\sigma_0|\,x^2}{v_0(l-x)}\right) \qquad (3.1)$$

where $H(\xi) \to 0$ when $\xi \to 0$. This estimate indicates, in particular, that when $l \to \infty$ the curve $x = x_0(t)$ approaches the x-axis.

In the following we stipulate that $x_0'(t) > 0$ when $t \leqslant T$ and assume that $t < T$ everywhere. For convenience we shall also assume that $\mu = 1$ and $\rho_0 = 1$. With these assumptions, Equations (1.2) and (1.3) can be transformed into the heat conduction equation

$$v_t = v_{xx} \qquad (3.2)$$

without affecting the remaining assumptions of Section 1.

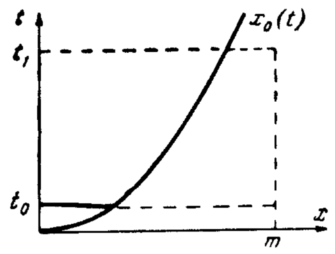

Fig. 2

To solve (3.1) we shall construct in the x, t plane a rectangle $t_1 \geqslant t \geqslant t_0$, $m \geqslant x \geqslant 0$, within which $m > x_0(t_1)$, $T > t_1 > t_0 > 0$, and outside which t_0, t_1 and m are arbitrary (Fig. 2). We shall introduce into this rectangle an auxiliary function $u^m(t,x)$ which satisfies (3.2) within the rectangle and the following conditions on its boundary:

$$u^m(t,0) = v_0, \quad u^m(t_0,x) = 0, \quad u_x^{\,m}(t,m) = 0 \qquad (3.3)$$

Function $u^m(t,x)$ can easily be written in an explicit form, but we shall need to refer to only some of its properties which can easily be proved to be

$$u_x^{\,m} < 0 \ \text{for} \ x < m, \qquad u^m(t,m) = u^1\left(\frac{t}{m^2},\,1\right) \equiv v_0 h\left(\frac{t-t_0}{m^2}\right) \qquad (3.4)$$

where $h(\xi)$ is some function independent of the parameters of the problem; it can be shown that $\xi^{-1}h(\xi) \to 0$ when $\xi \to 0$. We shall prove that $v(t,x_0(t)) > u^m(t,m)$. For this purpose we shall introduce function

$$w(t,x) = v(t,x) - u^m(t_0,x).$$

Within the quadrangle D_3 bounded by lines $t = t_0$, $t = t_1$, $x = 0$ and by curve $x = x_0(t)$, function w satisfies (3.2) and the following conditions:

$$w(t,0) = 0, \qquad w(t_0,x) = v(t_0,x), \qquad w_x(t,x_0(t)) = -u_x^{\,m}(t,x_0(t)) \qquad (3.5)$$

According to the theorem of maxima, function w has a minimum either at

$x = 0$ or on $t = t_0$, or on $x = x_0(t)$. However, by virtue of (3.5) function w cannot have a minimum on the curve $x = x_0(t)$, since in that case one would have at the point of munimum $w_x \leqslant 0$. Since $w = 0$ at $x = 0$ and $w \geqslant 0$ at $t = t_0$, then $w \geqslant 0$ everywhere within D_3, and in particular, $w(t, x_0(t)) \geqslant 0$. Whence we conclude, bearing in mind that $u_x^* < 0$,

$$v(t, x_0(t)) \geqslant u^m(t, x_0(t)) > u^m(t, m) = v_0 h\left(\frac{t - t_0}{m^2}\right) \tag{3.6}$$

Since this equation holds for any $t_0 > 0$, $m > x_0(t_1)$, and since $h(\xi)$ is continuous, it follows that (3.6) holds also for $t_0 = 0$, $m = x_0(t_1)$. From (1.5) we have

$$v(t_1, x_0(t_1)) = \int_0^{t_1} \frac{|\sigma_0|}{l - x_0(\xi)} d\xi \leqslant \frac{|\sigma_0|}{l - x_0(t_1)} t_1 \tag{3.7}$$

From (3.6) and (3.7) we obtain (writing t for t_1, since t_1 can be chosen arbitrarily)

$$\frac{|\sigma_0|}{l - x_0(t)} t \geqslant v_0 h\left(\frac{t}{x_0^2(t)}\right) \tag{3.8}$$

or

$$\frac{|\sigma_0|}{v_0(l - x_0)} x_0^2 \geqslant \frac{x_0^2}{t} h\left(\frac{t}{x_0^2}\right) \tag{3.9}$$

We shall introduce function $H(\xi)$ defined by Equation

$$H(\xi^{-1} h(\xi)) = \xi$$

Function $\xi^{-1}h(\xi)$ tends to zero when $\xi \to 0$, and there exists $\beta > 0$, such that when $\xi < \beta$ function $\xi^{-1}h(\xi)$ increases monotonously; therefore $H(0) = 0$ and $H(\xi)$ is determinate and monotonous for $\xi < \beta^{-1}h(\beta)$. From (3.9) we have

$$\tau(x) \leqslant x^2 H\left(\frac{|\sigma_0| x^2}{v_0(l - x)}\right) \tag{3.10}$$

It follows from this equation that when $l \to \infty$ then $\tau(x) \to 0$ for any fixed value of x.

It also follows from (3.10) that if

$$x_0 = A(t)\sqrt{t}$$

when $t \to 0$, then $A(t) \to \infty$ when $t \to 0$.

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BIBLIOGRAPHY

1. Barenblatt, G.I. and Ishlinskii, A.Iu., Ob udare viazko-plasticheskogo sterzhnia o zhestkuiu pregradu (On the impact of a visco-plastic bar on a rigid obstacle). *PMM* Vol.26, № 3, 1962.

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